ch: set & Seq (part 4)

**Theorem**: Every convergent seq is bounded

[H.W: show converse is false]

**proof**: $\forall \varepsilon > 0 \ \exists n_0 \ni |x_n - l| < \varepsilon \Rightarrow l - \varepsilon < x_n < l + \varepsilon \ \ \forall n \geq n_0$

Let $m = \min\{x_1, x_2, \ldots, x_{n_0-1}, l-\varepsilon\}$
$M = \max\{x_1, x_2, \ldots, x_{n_0-1}, l+\varepsilon\}$ $\Rightarrow m \leq x_n \leq M \ \ \forall n \in \mathbb{N}$

**Theorem**: $\{x_n\} \to l$ and $\{y_n\} \to m$ and $y_n > x_n \ \forall n$. Then $m \geq l$

(H.W.) Write down by yourselves

**Theorem:** Let $\{x_n\} \to l$, $\{y_n\} \to m$

(a) $\{c x_n\} \longrightarrow c \cdot l$

(b) $\{x_n \pm y_n\} \longrightarrow l \pm m$

(c) $\{x_n \cdot y_n\} \longrightarrow l \cdot m$

(d) $\left\{\dfrac{x_n}{y_n}\right\} \longrightarrow \dfrac{l}{m}$ if $m \neq 0$

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H.W. (1) Show $\{|x_n|\} \to |l|$. Give counterexam[ple] that converse is false

(2) Show $\{|x_n \pm y_n|\} \longrightarrow |l \pm m|$

(3) Show $\left\{\frac{1}{n}\cos\left(\frac{n\pi}{4}\right)\right\} \longrightarrow 0$

# Sandwitch theorem

If $\{x_n\}, \{y_n\}, \{z_n\}$ be three seq s.t. $x_n \le y_n \le z_n$. If $\{x_n\}, \{z_n\}$ both conv. to same limit then $\{y_n\}$ will also conv. to that lim

ie If $\{x_n\} \to l$, $\{z_n\} \to l$ then $\{y_n\} \to l$

**proof** :- [H. W. Do details from any text book]

**Hint** : $|x_n - l| < \varepsilon \ \forall n \geq n_1 \Rightarrow l - \varepsilon < x_n < l + \varepsilon \ \forall n \geq n_1$

$|z_n - l| < \varepsilon \ \forall n \geq n_2 \Rightarrow l - \varepsilon < z_n < l + \varepsilon \ \forall n \geq n_2$

Take $n_3 = \max\{n_1, n_2\}$ & use above eqn & $x_n \le y_n \le z_n$ to get

$$l - \varepsilon < x_n \le y_n \le z_n < l + \varepsilon \quad \forall n \geq n_3$$

$$\Rightarrow |y_n - l| < \varepsilon \quad \forall n \geq n_3 \quad \text{(Proved)}$$

H.W.

Show that $\{x_n\} \to 0$ where

$$\frac{n}{n^2} = \frac{1}{n} \downarrow 0$$

$$x_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$$

Solution: $x_n = \sum_{r=1}^{n} \frac{1}{(n+r)^2}$

$$\sum_{r=1}^{n} \frac{1}{(n+n)^2} \le x_n \le \sum_{r=1}^{n} \frac{1}{(n+1)^2}$$

$$\Rightarrow \frac{n}{4n^2} \le x_n \le \frac{n}{(n+1)^2} \Rightarrow \frac{1}{4n} \le x_n \le \frac{1}{n}$$

Now, $\{\frac{1}{n}\} \to 0$, $\{\frac{1}{4n}\} \to 0$

Hence, by sandwitch thm, $\{x_n\} \to 0$

H. W: show the following

(1) $\lim_{n\to\infty}\left[\frac{1}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\cdots+\frac{n^2}{n^3}\right]=0$

(2) $\lim_{n\to\infty}\left[3^n+4^n\right]^{1/n}=4$

$4\left(\left(\frac{3}{4}\right)^n+1\right)^{\frac{1}{n}}$ $\lim\left[\left(1+f^{n}\right)^{\frac{1}{n}}\right]=1^0$

(3) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

$\infty-\infty=0$ ✗

$\frac{1}{\sqrt{n}+\sqrt{n+1}}$

$\lim_{n\to\infty}\sqrt{n}=\infty$

$\lim a_n b_n = (\lim a_n)(\lim b_n)$

A seq is called null seq if $\lim x_n=0$

V. Imp [Theorem on Null seq]

**Theorem 1** : Let $\{x_n\}$ be a seq of positive real numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = m$. If $0 \le m < 1$ then $\lim_{n\to\infty} x_n = 0$

**Theorem 2** : Let $\{x_n\}$ be a seq of positive real numbers such that $x_n^{1/n} = m$. If $0 \le m < 1$ then $\lim x_n = 0$

H.W: Show that:

$\frac{n!}{n^n} \to \frac{(n-1)^{n-1}}{(n-1)!} \to \frac{n}{(n-1)^{n-1}} \to 0$

(1) $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$

(2) $\lim\limits_{n\to\infty} \frac{4^{3n}}{3^{4n}} = 0$

(3) $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a>1)$

(4) $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0$

(5) $\lim\limits_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$

$[a, p > 0]$

$\frac{7^n}{n!} = \frac{7\times 7\times \cdots \times 7}{1\times 2\times \cdots \times n}$

$1000$

## (V. Imp) Limit Theorems of Cauchy

(1st Limit Theorem) If $\{x_n\} \to m$,

then $\left\{ \dfrac{x_1 + x_2 + \cdots + x_n}{n} \right\} \to m$

★

(2nd Limit Theorem) Let $x_n > 0 \ \forall n \in \mathbb{N}$

and $\lim\limits_{n \to \infty} \dfrac{x_{n+1}}{x_n} = m$

Then $\boxed{\lim\limits_{n \to \infty} (x_n)^{1/n} = m}$

[Trick: Take $y_n = x_n^{1/n}$ where $y_n$ is of interest]

## Examples (v.gmp)

(1) Show that, $\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1$

Hint: Let $x_n = \sqrt[n]{n} \Rightarrow \lim x_n = 1$

Cauchy 1st Limit $\Rightarrow \lim \frac{x_1+x_2+\cdots+x_n}{n} = 1$

(2) If $x_n > 0 \ \forall n \in \mathbb{N}$ and $\lim x_n = m \ (\neq 0)$

then $\lim_{n\to\infty} (x_1 \cdot x_2 \cdots x_n)^{1/n} = m$

Hint: $\lim(\log(*)) = \log(\lim(*))$

**H.W:** Find the following Limits [Hint: Cauchy 1st limit]

(1) $\lim\limits_{n \to \infty} \frac{1}{n}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right] = 0$

(2) $\lim\limits_{n \to \infty} \frac{1}{2n+1}\left[1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}\right]$

(3) $\lim\limits_{n \to \infty} \frac{1}{n}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right]$

(4) $\lim\limits_{n \to \infty} n\left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right]$

**Ans:** (1) 0 | (2) 0 | (3) 0 | (4) 1

V. Imp

$\frac{1}{n}$ Example (Application of Cauchy 2nd limit Thm)

(1) Find the limit $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$ [Ans $\frac{1}{e}$]

$\frac{1}{n}\left(\frac{1}{n}\sum \log n\right)$

Ans: Let $x_n = \frac{n!}{n^n}$ Then,

$$\lim \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{\lim\left(1+\frac{1}{n}\right)^n}$$

$$= 1/e$$

Hence $\lim x_n = 1/e$

H.W: Find the Limits [Hint: Apply Cauchy 2nd lim]

(1) $\lim_{n\to\infty} \frac{1}{n}\left[(2n+1)(2n+2)\cdots(2n+n)\right]^{1/n}$

(2) $\lim_{n\to\infty} n^{1/n}$

(3) $\lim_{n\to\infty} \frac{1}{n}\left[(n+1)(n+2)\cdots(2n)\right]^{1/n}$

(4) $\lim_{n\to\infty} \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right]^{1/n}$

| Ans: (1) $\frac{27}{4e}$ | (2) $1$ | (3) $\frac{4}{e}$ | (4) $e$ |
|---|---|---|---|

# Some Imp Inequality Results

(a) $n^{\frac{1}{n}}$ is monotonically decreasing $(n \geq 3)$

Hint: To show: $(n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \Leftrightarrow \left(1+\frac{1}{n}\right)^n < n$

Now observe* $\left(1+\frac{1}{n}\right)^n \leq e \leq n$ [for $n \geq 3$]

*proved later

$$\Rightarrow \left(1+\frac{1}{n}\right) \leq e^{\frac{1}{n}} \leq n^{\frac{1}{n}} \text{ [proved]}$$

H.W.

(b) $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$

Take $\left(1-\frac{1}{n}\right), \underbrace{1, 1, \ldots, 1}_{n \text{ times}}$, apply AM > GM

V. Imp

H.W.

(c)

$$(n+2)^{n+1} < (n+1)^{n+2}$$

Hint: From (a),

$$(n+2)^{\frac{1}{n+2}} < (n+1)^{\frac{1}{n+1}}$$

# Monotone seq

(1) $\{x_n\}$ increasing if $x_{n+1} \geq x_n \; \forall n$

(2) $\{x_n\}$ strictly " if $x_{n+1} > x_n \; \forall n$

(3) $\{x_n\}$ decreasing if $x_{n+1} \leq x_n \; \forall n$

(4) $\{x_n\}$ strictly " if $x_{n+1} < x_n \; \forall n$

(5) $\{x_n\}$ monotone if either increasing or decreasing

H.W. Check for monotone seq

(1) $\{n\}$

(2) $\{n^2\}$

(3) $\left\{\frac{1}{n}\right\}$

(4) $\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}$

* (5) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

* (6) $\{n^{1/n}\}$

see previous pages, Already done!

☆ Imp

(1) A monotone increasing seq & bdd above is convergent to its sup. $-\frac{1}{n} \uparrow 0$

(2) A monotone decreasing seq & bdd below is convergent to its inf. $\frac{1}{n} \downarrow 0$

☆ Ex: Show $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent

[ ie $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists ]

[strictly $\uparrow$, bdd above by ☆ (This proof is done in next to next slide)

$\underbrace{\text{strictly}}_{\text{done in previous slide}}$

v. Imp

Ex 1

$x_1 = \sqrt{2}$

$x_n = \sqrt{2 + x_{n-1}}$

Find $\lim_{n \to \infty} x_n$

Hint

Show $x_n \uparrow$, show $0 < x_n \leq 2$ [by induction]

$\Rightarrow x_n \to \lambda$ for some $\lambda$

$\Rightarrow \lambda = \sqrt{2 + \lambda} \Rightarrow \lambda^2 = 2 + \lambda \Rightarrow \lambda = 2$

Ex 2 Let $x_1, x_2 > 0$. $\{x_n\}$ is defined by

H.W.

$x_{n+2} = \sqrt{x_{n+1} \cdot x_n}$. Show $x_n \longrightarrow (x_1 x_2^2)^{1/3}$

Hint: Monotone $\uparrow$, bdd above by $x_1$, bdd below by $x_2$.

Very imp

## ★ Proof of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists & Euler number (e)

**Step 1** Expand $\left(1+\frac{1}{n}\right)^n$ by Binomial Thm :

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$\lim \left(1+\frac{1}{n}\right)^n = e$

**Step 2** : Show : $\binom{n}{k}\left(\frac{1}{n}\right)^k \le \frac{1}{k!} \le \frac{1}{2^{k-1}}$ for $1 \le k \le n$

$e < 3$

$2 < e < 3$

$$\Rightarrow \sum_{k=1}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \le \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \le 2$$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \le 1 + 2$$

[Hence, bdd above by 3]

<u>Step 3</u> : Expand and observe/ verify ;

$$\binom{n+1}{k}\left(\frac{1}{n+1}\right)^{k} \geq \binom{n}{k}\left(\frac{1}{n}\right)^{k}$$

$$\Rightarrow \sum \binom{n+1}{k}\left(\frac{1}{n+1}\right)^{k} \geq \sum \binom{n}{k}\left(\frac{1}{n}\right)^{k}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^{n}$$

★ [Monotone increasing]

Hence, convergent [lim exist & called e [Euler number]]